SET
&
RELATIONS

(KEY CONCEPTS + SOLVED EXAMPLES)

SET & RELATIONS

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KEY CONCEPTS

1. Set Theory

Collection of well defined objects which are distinct and distinguishable. A collection is said to be well defined if each and every element of the collection has some definition.

1.1 Notation of a set : Sets are denoted by capital letters like A, B, C or { } and the entries within the bracket are known as elements of set.

1.2 Cardinal number of a set : Cardinal number of a set X is the number of elements of a set X and it is denoted by n(X) e.g. $X = [x_1, x_2, x_3]$ $\therefore$ n(X) =3

2. Representation of Sets

2.1 Set Listing Method (Roster Method) :

In this method a set is described by listing all the elements, separated by commas, within braces { }

2.2 Set builder Method (Set Rule Method) :

In this method, a set is described by characterizing property P(x) of its elements x. In such case the set is described by {x : P(x) holds} or {x | P(x) holds}, which is read as the set of all x such that P(x) holds. The symbol '|' or '**:**' is read as such that.

3. Type of Sets

3.1 Finite set :

A set X is called a finite set if its element can be listed by counting or labeling with the help of natural numbers and the process terminates at a certain natural number n. i.e. n(X) = finite no. eg (a) A set of English Alphabets (b) Set of soldiers in Indian Army

3.2 Infinite set :

A set whose elements cannot be listed counted by the natural numbers (1, 2, 3.......n) for any number n, is called a infinite set. e.g.

(a) A set of all points in a plane

(b) $X = \{x : x \in R, 0 < x < 0.0001\}$

(c) $X = \{x : x \in Q, 0 \le x \le 0.0001\}$

3.3 Singleton set :

A set consisting of a single element is called a singleton set. i.e. n (X) = 1,

e.g. $\{x : x \in N, 1 < x < 3\}$, {{}} : Set of null set, $\{\phi\}$ is a set containing alphabet ϕ.

3.4 Null set :

A set is said to be empty, void or null set if it has no element in it, and it is denoted by ϕ. i.e. X is a null set if n(X) = 0. e.g. : $\{x : x \in R \text{ and } x^2 + 2 = 0\}, \{x : x > 1 \text{ but } x < 1/2\}, \{x : x \in R, x^2 < 0\}$

3.5 Equivalent Set :

Two finite sets A and B are equivalent if their cardinal numbers are same i.e. n(A) = n(B).

3.6 Equal Set :

Two sets A and B are said to be equal if every element of A is a member of B and every element of B is a member of A. i.e. A = B, if A and B are equal and $A \ne B$, if they are not equal.

4. Universal Set

It is a set which includes all the sets under considerations i.e. it is a super set of each of the given set. Thus, a set that contains all sets in a given context is called the universal set. It is denoted by U. e.g. If A = {1, 2, 3}, B = {2, 4, 5, 6} and C = {1, 3, 5, 7}, then U = {1, 2, 3, 4, 5, 6, 7} can be taken as the universal set.

5. Disjoint Set

Sets A and B are said to be disjoint iff A and B have no common element or $A \cap B = \phi$. If $A \cap B \neq \phi$ then A and B are said to be intersecting or overlapping sets.

e.g. : (i) If A = {1, 2, 3}, B = {4, 5, 6} and C = {4, 7, 9} then A and B are disjoint set where B and C are intersecting sets. **(ii)** Set of even natural numbers and odd natural numbers are disjoint sets.

6. Complementary Set

Complementary set of a set A is a set containing all those elements of universal set which are not in A. It is denoted by $\overline{A}$, A^C or A'. So $A^C = \{x : x \in U \text{ but } x \notin A\}$. e.g. If set A = {1, 2, 3, 4, 5} and universal set U = {1, 2, 3, 4,.......50} then $\overline{A}$ = {6, 7, ...50}

NOTE :

All disjoint sets are not complementary sets but all complementary sets are disjoint.

7. Subset

A set A is said to be a subset of B if all the elements of A are present in B and is denoted by $A \subset B$ (read as A is subset of B) and symbolically written as : $x \in A \Rightarrow x \in B \Leftrightarrow A \subset B$

7.1 Number of subsets :

Consider a set X containing n elements as $\{x_1, x_2, ..., x_n\}$ then the total number of subsets of X = 2^n

Proof : Number of subsets of above set is equal to the number of selections of elements taking any number of them at a time out of the total n elements and it is equal to 2^n

$\because \quad {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

7.2 Types of Subsets :

A set A is said to be a **proper subset** of a set B if every element of A is an element of B and B has at least one element which is not an element of A and is denoted by $A \subset B$.

The set A itself and the empty set is known as **improper subset** and is denoted as $A \subseteq B$.

e.g. If X = $\{x_1, x_2, ...,x_n\}$ then total number of proper sets = $2^n - 2$ (excluding itself and the null set). The statement $A \subset B$ can be written as $B \supset A$, then B is called the **super set** of A and is written as $B \supset A$.

8. Power Sets

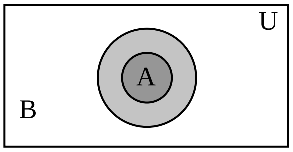
The collection of all subsets of set A is called the power set of A and is denoted by P(A)

i.e. P(A) = {x : x is a subset of A}. If X = $\{x_1, x_2, x_3, ...x_n\}$ then $n(P(X)) = 2^n$; $n(P(P(x))) = 2^{2^n}$.

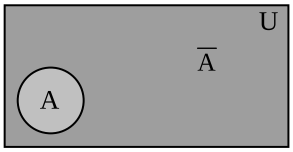
9. Venn(Euler) Diagrams

The diagrams drawn to represent sets are called Venn diagram or Euler-Venn diagrams. Here we represents the universal U as set of all points within rectangle and the subset A of the set U is represented by the interior of a circle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B by two intersecting circles.

e.g. If A is subset of B then it is represented diagrammatically in fig.



e.g. If A is a set then the complement of A is represented in fig.



10. Operations on Sets

10.1 Union of sets :

If A and B are two sets then union ($\cup$) of A and B is the set of all those elements which belong either to A or to B or to both A and B. It is also defined as $A \cup B = \{x : x \in A \text{ or } x \in B\}$. It is represented through Venn diagram in fig.1 & fig.2

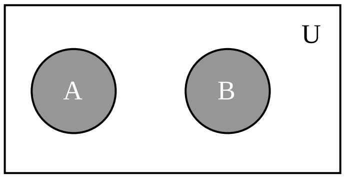


Fig. (1)

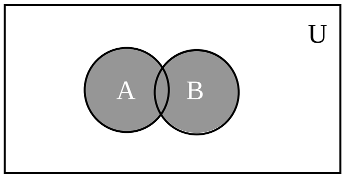
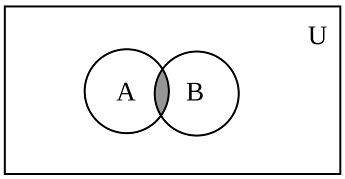


Fig. (2)

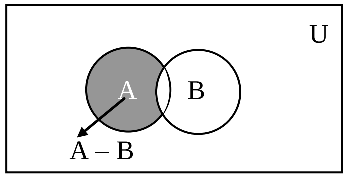
10.2 Intersection of sets :

If A and B are two sets then intersection ($\cap$) of A and B is the set of all those elements which belong to both A and B. It is also defined as $A \cap B = \{x : x \in A \text{ and } x \in B\}$ represented in Venn diagram (see fig.)



10.3 Difference of two sets :

If A and B are two sets then the difference of A and B, is the set of all those elements of A which do not belong to B.



Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

or $A - B = \{x \in A ; x \notin B\}$

Clearly $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

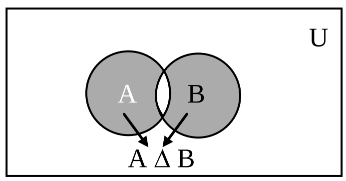
It is represented through the Venn diagrams.

10.4 Symmetric difference of two sets :

Set of those elements which are obtained by taking the union of the difference of A & B is (A – B) & the difference of B & A is (B – A), is known as the symmetric difference of two sets A & B and it is denoted by $(A \Delta B)$.

Thus $A \Delta B = (A - B) \cup (B - A)$

Representation through the venn diagram is given in the fig.



11. Number of Elements in Different Sets

If A, B & C are finite sets and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B)$ (if A & B are disjoint sets)

(iii) $n(A - B) = n(A) - n(A \cap B)$

(iv) $n(A \Delta B) = n[(A - B) \cup (B - A)]$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$

$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(vii) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

12. Cartesian Product of two Sets

Cartesian product of A to B is a set containing the elements in the form of ordered pair (a, b) such that $a \in A$ and $b \in B$. It is denoted by $A \times B$.

i.e. $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

$= \{(a_1, b_1),(a_1, b_2),(a_2, b_1),(a_2, b_2),(a_3, b_1),(a_3, b_2)\}$

If set $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$ then

$A \times B$ and $B \times A$ can be written as :

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ and

$B \times A = \{(b, a) ; b \in B \text{ and } a \in A\}$

Clearly $A \times B \neq B \times A$ until A and B are equal

Note :

1. If number of elements in A : $n(A) = m$ and $n(B) = n$ then number of elements in $(A \times B) = m \times n$
2. Since $A \times B$ contains all such ordered pairs of the type (a, b) such that $a \in A$ & $b \in B$, that means it includes all possibilities in which the elements of set A can be related with the elements of set B. Therefore, $A \times B$ is termed as largest possible relation defined from set A to set B, also known as universal relation from A to B.

13. Algebraic Operations on Sets

13.1 Idempotent operation :

For any set A, we have (i) $A \cup A = A$ and (ii) $A \cap A = A$

Proof :

(i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(ii) $A \cap A = \{x : x \in A \;\&\; x \in A\} = \{x : x \in A\} = A$

13.2 Identity operation :

For any set A, we have

(i) $A \cup \phi = A$ and

(ii) $A \cap U = A$ i.e. ϕ and U are identity elements for union and intersection respectively

Proof :

(i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\}$

$= \{x : x \in A\} = A$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\}$

$= \{x : x \in A\} = A$

13.3 Commutative operation :

For any set A and B, we have

(i) $A \cup B = B \cup A$ and (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

13.4 Associative operation :

If A, B and C are any three sets then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

i.e. union and intersection are associative.

13.5 Distributive operation :

If A, B and C are any three sets then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union respectively.

13.6 De-Morgan's Principle :

If A and B are any two sets, then

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Proof : (i) Let x be an arbitrary element of $(A \cup B)'$. Then $x \in (A \cup B)'$ $\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$ $\Rightarrow x \in A' \cap B'$

Again let y be an arbitrary element of $A' \cap B'$. Then $y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$ $\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$ $\Rightarrow y \in (A \cup B)'$

$\therefore A' \cap B' \subseteq (A \cup B)'$.

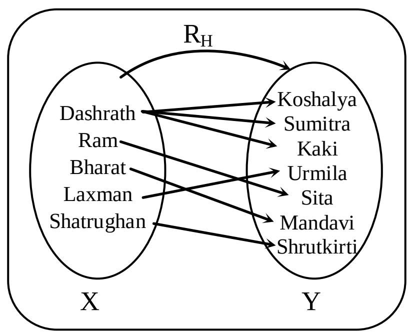
Hence $(A \cup B)' = A' \cap B'$

Similarly (ii) can be proved.

14. Relation

A relation R from set X to Y ($R : X \to Y$) is a correspondence between set X to set Y by which some or more elements of X are associated with some or more elements of Y. Therefore a relation (or binary relation) R, from a non-empty set X to another non-empty set Y, is a subset of $X \times Y$. i.e. $R_H : X \to Y$ is nothing but subset of $A \times B$.

e.g. Consider a set X and Y as set of all males and females members of a royal family of the kingdom Ayodhya X = {Dashrath, Ram, Bharat, Laxman, shatrughan} and Y = {Koshaliya, Kakai, sumitra, Sita, Mandavi, Urmila, Shrutkirti} and a relation R is defined as "was husband of "from set X to set Y.



Then R_H = {(Dashrath, Koshaliya), (Ram, sita), (Bharat, Mandavi), (Laxman, Urmila), (Shatrughan, Shrutkirti), (Dashrath, Kakai), (Dashrath, Sumitra)}

Note :

(i) If a is related to b then symbolically it is written as a R b where a is pre-image and b is image

(ii) If a is not related to b then symbolically it is written as $a \not R\ b$.

14.1 Domain, Co-domain & Range of Relation :

Domain : of relation is collection of elements of the first set which are participating in the correspondence i.e. it is set of all pre-images under the relation R. e.g. Domain of R_H : {Dashrath, Ram, Bharat, Laxman, Shatrughan}

Co-Domain : All elements of set Y irrespective of whether they are related with any element of X or not constitute co-domain. e.g. Y = {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti} is co-domain of R_H.

Range : of relation is a set of those elements of set Y which are participating in correspondence i.e. set of all images. Range of R_H : {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti}.

15. Types of Relations

15.1 Reflexive Relation

$R : X \to Y$ is said to be reflexive iff $x R x$ $\forall x \in X$. i.e. every element in set X, must be a related to itself therefore $\forall x \in X$; $(x, x) \in R$ then relation R is called as reflexive relation.

15.2 Identity Relation :

Let X be a set. Then the relation $I_x = \{(x, x) : x \in X\}$ on X is called the identity relation on X. i.e. a relation I_x on X is identity relation if every element of X related to itself only. e.g. $y = x$

Note : All identity relations are reflexive but all reflexive relations are not identity.

15.3 Symmetric Relation

$R : X \to Y$ is said to be symmetric iff $(x, y) \in R \Rightarrow (y, x) \in R$ for all $(x, y) \in R$ i.e. $x R y \Rightarrow y R x$ for all $(x, y) \in R$. e.g. perpendicularity of lines in a plane is symmetric relation.

15.4 Transitive Relation

$R : X \to Y$ is transitive iff $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ for all (x, y) and $(y, z) \in R$. i.e. $x R y$ and $y R z \Rightarrow x R z$. e.g. The relation "being sister of " among the members of a family is always transitive.

Note :

(i) Every null relation is a symmetric and transitive relation.

(ii) Every singleton relation is a transitive relation.

(iii) Universal and identity relations are reflexive, symmetric as well as transitive.

15.5 Anti-symmetric Relation

Let A be any set. A relation R on set A is said to be an antisymmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$ e.g. Relations "being subset of "; "is greater than or equal to" and "identity relation on any set A" are antisymmetric relations.

15.6 Equivalence Relation

A relation R from a set X to set Y ($R : X \to Y$) is said to be an equivalence relation iff it is reflexive, symmetric as well as transitive. The equivalence relation is denoted by ~ e.g. Relation "is equal to" Equality, Similarity and congruence of triangles, parallelism of lines are equivalence relation.

16. Inverse of a Relation

Let A, B be two sets and let R be a relation from a set A to B. Then the inverse of R, denoted by R^{-1} , is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$, Clearly,

$(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$ Also,

Dom of R = Range of R^{-1} and

Range of R = Dom of R^{-1}

SOLVED EXAMPLES

Ex.1 If a set A = {a, b, c} then find the number of subsets of the set A and also mention the set of all the subsets of A.

Sol. Since n(A) = 3

$\therefore$ number of subsets of A is $2^3 = 8$

and set of all those subsets is P(A) named as power set

P(A):{ϕ,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}

Ex.2 Show that n {P[P(ϕ)]} = 4

Sol. We have P(ϕ) = {ϕ} $\quad\therefore$ P(P(ϕ)) = {ϕ, {ϕ}}

$\Rightarrow$ P[P(P(ϕ))] = { ϕ, {ϕ}, {{ϕ}}, {ϕ, {ϕ}}}.

Hence, n{P[P(ϕ)]} = 4

Ex.3 If A = {x : x = 2n + 1, n $\in$ Z and B = {x : x = 2n, n $\in$ Z}, then find A $\cup$ B.

Sol. A $\bigcup$ B = {x : x is an odd integer} $\cup$ {x : x is an even integer} = {x : x is an integer} = Z

Ex.4 If A = {x : x = 3n, n $\in$ Z} and B = {x : x = 4n, n $\in$ Z} then find A $\cap$ B.

Sol. We have,

x $\in$ A $\cap$ B $\Leftrightarrow$ x = 3n, n $\in$ Z and x = 4n, n $\in$ Z

$\Leftrightarrow$ x is a multiple of 3 and x is a multiple of 4

$\Leftrightarrow$ x is a multiple of 3 and 4 both

$\Leftrightarrow$ x is a multiple of 12 $\Leftrightarrow$ x = 12n, n $\in$ Z

Hence A $\cap$ B = {x : x = 12n, n $\in$ Z}

Ex.5 If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in A $\cup$ B? Find also, the maximum number of elements in A $\cup$ B.

Sol. We have, n(A $\cup$ B) = n(A) + n(B) – n(A $\cap$ B).

This shows that n(A $\cup$ B) is minimum or maximum according as n(A $\cap$ B) is maximum or minimum respectively.

Case-I

When n(A $\cap$ B) is minimum, i.e., n(A $\cap$ B) = 0

This is possible only when A $\cap$ B = ϕ.

In this case,

n(A$\cup$B) = n(A)+ n(B) – 0=n(A)+n(B)=3+6= 9.

So, maximum number of elements in A$\cup$ B is 9.

Case-II

When n(A $\cap$ B) is maximum.

This is possible only when A $\subseteq$ B. In this case, n(A $\cap$ B) = 3

$\therefore$ n(A $\cup$ B) = n(A) + n(B) – n(A $\cap$ B)

= (3 + 6 – 3) = 6

So, minimum number of elements in A $\cup$ B is 6.

Ex.6 If A = {2, 3, 4, 5, 6, 7} and B = {3, 5, 7, 9, 11, 13} then find A – B and B – A.

Sol. A – B = {2, 4, 6} & B – A = {9, 11, 13)

Ex.7 If the number of elements in A is m and number of element in B is n then find

(i) The number of elements in the power set of A × B.

(ii) number of relation defined from A to B

Sol. **(i)** Since n(A) = m; n(B) = n

then n(A × B)= mn

So number of subsets of A × B = 2^{mn}

$\Rightarrow$ n (P(A × B)) = 2^{mn}

(ii) number of relation defined from A to B = 2^{mn}

Any relation which can be defined from set A to set B will be subset of A × B

$\because$ A × B is largest possible relation A $\to$ B

$\therefore$ no. of relation from A $\to$ B = no. of subsets of set (A × B)

Ex.8 Let A and B be two non-empty sets having elements in common, then prove that A × B and B × A have n^2 elements in common.

Sol. We have (A × B) $\cap$ (C × D) = (A $\cap$ C)× (B $\cap$ D)

On replacing C by B and D by A, we get

$\Rightarrow$ (A × B) $\cap$ (B × A) = (A $\cap$ B) × (B $\cap$ A)

It is given that AB has n elements so (A $\cap$ B) × (B $\cap$ A) has n^2 elements

But $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

$\therefore$ $(A \times B) \cap (B \times A)$ has n^2 elements

Hence $A \times B$ and $B \times A$ have n^2 elements in common.

Ex.9 Let R be the relation on the set N of natural numbers defined by

R : {(x, y)} : x + 3y = 12 $x \in N, y \in N$} Find

(i) R **(ii)** Domain of R

(iii) Range of R

Sol. **(i)** We have, $x + 3y = 12 \Rightarrow x = 12 - 3y$

Putting y = 1, 2, 3, we get x = 9, 6, 3 respectively

For y = 4, we get $x = 0 \notin N$. Also for $y > 4, x \notin N$

$\therefore$ R = {(9, 1), (6, 2), (3, 3)}

(ii) Domain of R = {9, 6, 3}

(iii) Range of R = {1, 2, 3}

Ex.10 If $X = \{x_1, x_2, x_3\}$ and $y = (x_1, x_2, x_3, x_4, x_5\}$ then find which is a reflexive relation of the following :

(a) $R_1 : \{(x_1, x_1), (x_2, x_2)$

(b) $R_1 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3)$

(c) $R_3 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_1, x_3), (x_2, x_4)$

(d) $R_3 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)$

Sol. (a) non-reflexive because $(x_3, x_3) \notin R_1$

(b) Reflexive

(c) Reflexive

(d) non-reflexive because $x_4 \notin X$

Ex.11 If x = {a, b, c} and y = {a, b, c, d, e, f} then find which of the following relation is symmetric relation :

R_1 : { } i.e. void relation

R_2 : {(a, b)}

R_3 : {(a, b), (b, a)(a, c)(c, a)(a, a)}

Sol. R_1 is symmetric relation because it has no element in it.

R_2 is not symmetric because $(b, a) \in R_2$ & R_3 is symmetric.

Ex.12 If x = {a, b, c} and y = (a, b, c, d, e} then which of the following are transitive relation.

(a) R_1 = { }

(b) R_2 = {(a, a)}

(c) R_3 = {(a, a}.(c, d)}

(d) R_4 = {(a, b), (b, c)(a, c),(a, a), (c, a)}

Sol. **(a)** R_1 is transitive relation because it is null relation.

(b) R_2 is transitive relation because all singleton relations are transitive.

(c) R_3 is transitive relation

(d) R_4 is also transitive relation

Ex.13 Let R be a relation on the set N of natural numbers defined by $xRy \Leftrightarrow x$ divides y′ for all x, $y \in N$.

Sol. This relation is an antisymmetric relation on set N. Since for any two numbers $a, b \in N$. $a \mid b$ and $b \mid a \Rightarrow a = b$, i.e., a R b and b R a $\Rightarrow a = b$.

It should be noted that this relation is not antisymmetric on the set Z of integers, because we find that for any non zero integer a a R (a) and (–a) R a, but $a \neq - a$.

Ex.14 Prove that the relation R on the set Z of all integers numbers defined by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n is an equivalence relation on Z.

Sol. We observe the following properties

Reflexivity :

For any $a \in N$, we have

$a - a = 0 \times n \Rightarrow a - a$ is divisible by $n \Rightarrow (a, a) \in R$

Thus (a, a) R for all Z. so, R is reflexive on Z.

Symmetry :

Let $(a, b) \in R$. Then $(a, b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow (a - b) = np$ for some $p \in Z \Rightarrow b - a = n(-p)$

$\Rightarrow b - a$ is divisible by n $\Rightarrow (b, a) \in R$

Thus $(a, b) \in R \Rightarrow (b, a) \in R$ for all a, b, $\in$ Z.

So R is symmetric on Z.

Transitivity :

Let a, b, $c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$.

Then $(a, b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow a - b = np$ for some $p \in Z$

$(b, c) \in R \Rightarrow (b - c)$ is divisible by n

$\Rightarrow b - c = np$ for some $q \in Z$

$\therefore$ $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ a – b = np and b – c = nq

$\Rightarrow$ (a – b) + (b – c) = np + nq

$\Rightarrow$ a – c = n(p + q)

$\Rightarrow$ a – c is divisible by n $\Rightarrow$ (a, c) $\in$ R

Thus (a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ Z. So R is transitive relation on Z.

Thus, R being reflexive, symmetric and transitive is an equivalence relation on Z.

Ex.15 Let a relation R_1 on the set R of real numbers be defined as (a, b) $\in R_1 \Leftrightarrow$ 1 + ab > 0 for all a, b $\in$ R. Show that R_1 is reflexive and symmetric but not transitive.

Sol. We observe the following properties :

Reflexivity :

Let a be an arbitrary element of R. Then

a $\in$ R $\Rightarrow$ 1 + a . a = 1 + a^2 > 0 $\Rightarrow$ (a, a) $\in R_1$

Thus (a, a) $\in R_1$ for all a $\in$ R. So R_1 is reflexive on R.

Symmetry :

Let (a, b) $\in$ R. Then

(a, b) $\in R_1 \Rightarrow$ 1 + ab > 0 $\Rightarrow$ 1 + ba > 0

$\Rightarrow$ (b, a) $\in R_1$

Thus (a, b) $\in R_1 \Rightarrow$ (b, a) $\in R_1$ for all a, b $\in$ R

So R_1 is symmetric on R

Transitive :

We observe that (1, 1/2) $\in R_1$ and (1/2, –1) $\in R_1$ but (1,–1) $\in R_1$ because 1 + 1 × (–1) = 0 $\not>$ 0. So R_1 is not transitive on R.

Ex.16 Let A be the set of first ten natural numbers and let R be a relation on A defined by (x, y) $\in$ R $\Leftrightarrow$ x + 2y = 10 i.e., R = {(x, y) : x $\in$ A, y $\in$ A and x + 2y = 10}. Express R and R^{-1} as sets of ordered pairs. Determine also :

(i) Domains of R and R^{-1}

(ii) Range of R and R^{-1}

Sol. We have (x, y) $\in$ R $\Leftrightarrow$ x + 2y = 10 $\Leftrightarrow$ y = x, y $\in$ A

where A = {1,2,3,4,5,6,7,8,9,10}

Now, x = 1 $\Rightarrow$ y = $\notin$ A

This shows that 1 is not related to any element in A. Similarly we can observe that 3, 5, 7, 9 and 10 are not related to any element of a under the defined relation. Further we find that

for x = 2, y = $\frac{10-2}{2}$ = 4 $\in$ A $\therefore$ (2, 4) $\in$ R

for x = 4, y = $\frac{10-4}{2}$ = 3 $\in$ A $\therefore$ (4, 3) $\in$ R

for x = 6, y = $\frac{10-6}{2}$ = 2 $\in$ A $\therefore$ (6, 2) $\in$ R

for x = 8, y = $\frac{10-8}{2}$ = 1 $\in$ A $\therefore$ (8, 1) $\in$ R

Thus R = {(2, 4), (4, 3), (6, 2), (8, 1)}

$\Rightarrow$ R^{-1} = {(4, 2), (3, 4), (2, 6), (1, 8)}

Clearly, Dom (R) = {2, 4, 6, 8} = Range (R^{-1})

and Range (R) = {4, 3, 2, 1} = Dom (R^{-1})